Cambridge Assessment Admissions Testing

Test of Mathematics for University Admission

Paper 1 2018 worked answers

Test of Mathematics for University Admission, 2018 Paper 1 Worked Solutions

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Contents

Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the 'Preparing for the test' section on the Admissions Testing website.

Question 1

We write the integrand (the thing being integrated) in terms of powers of x and split up the fraction in order to be able to compute the integral:

$$\begin{aligned}
\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x &= \int_1^4 \frac{3-2x}{x^{\frac{3}{2}}}\,\mathrm{d}x \\
&= \int_1^4 3x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}}\,\mathrm{d}x \\
&= \left[\frac{3x^{-\frac{1}{2}}}{(-\frac{1}{2})} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 \\
&= \left[-6x^{-\frac{1}{2}} - 4x^{\frac{1}{2}}\right]_1^4 \\
&= (-6 \times \tfrac{1}{2} - 4 \times 2) - (-6 \times 1 - 4 \times 1) \\
&= -11 - (-10) \\
&= -1
\end{aligned}$$

so the correct answer is D.

Question 2

One approach is to use the formulae for the sum of the first 5 and the first 8 terms (that is, $S_n = \frac{1}{2}n(2a + (n-1)d)$).

We have

$$S_5 = \tfrac{1}{2} \times 5(2a + 4d) = \tfrac{5}{2}(2a + 4d)$$
$$S_8 = \tfrac{1}{2} \times 8(2a + 7d) = 4(2a + 7d)$$

These are equal, so we have

$$\tfrac{5}{2}(2a + 4d) = 4(2a + 7d)$$

We want to rearrange this to get a formula in the form $a = \cdots$, so we multiply both sides by 2, expand the brackets and collect like terms:

$$\begin{aligned} & 5(2a + 4d) = 8(2a + 7d) \\ \iff & 10a + 20d = 16a + 56d \\ \iff & -6a = 36d \\ \iff & a = -6d \end{aligned}$$

so the answer is C.

An alternative approach is to note that the difference between the sum of the first 5 and the sum of the first 8 terms equals the sum of the 6th, 7th and 8th terms. (This is because $S_5 = u_1 + u_2 + \cdots + u_5$ while $S_8 = u_1 + u_2 + \cdots + u_5 + u_6 + u_7 + u_8$, where we have written the sequence as u_1, u_2, $\ldots$.) Since the two sums S_5 and S_8 are equal, the sum of the 6th, 7th and 8th terms must be zero. This gives

$$(a + 5d) + (a + 6d) + (a + 7d) = 0$$

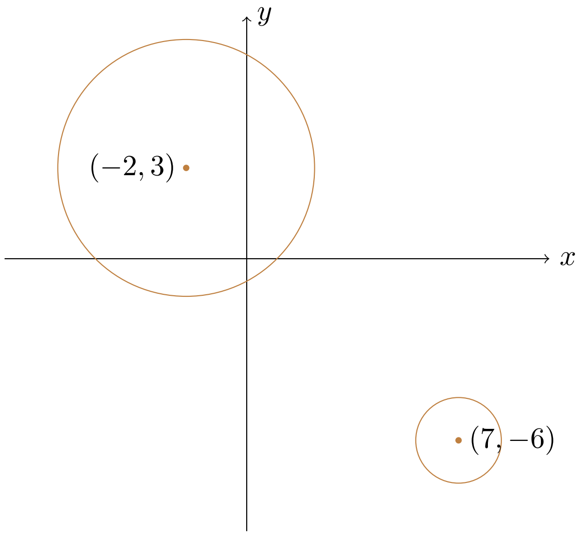
Expanding and collecting like terms gives $3a + 18d = 0$, so $a = -6d$, as before.

Question 3

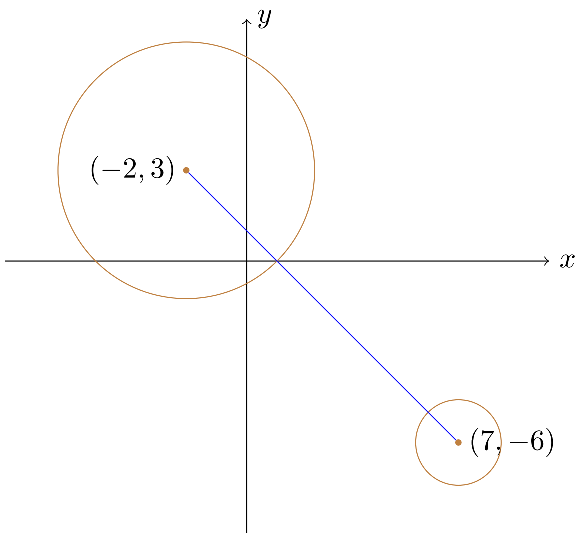
The first circle has centre $(-2, 3)$ and radius $\sqrt{18} = 3\sqrt{2}$

The second circle has centre $(7, -6)$ and radius $\sqrt{2}$

It would help to draw a quick sketch of the situation.



It seems clear from the sketch that to find the shortest distance between the circles, we should draw a line between their centres and find the length of the segment which lies outside both of them:



The distance between the centres is

$$\sqrt{((-2) - 7)^2 + (3 - (-6))^2} = \sqrt{81 + 81} = 9\sqrt{2}$$

and if we subtract the two radii ($3\sqrt{2}$ and $\sqrt{2}$) for the parts of the line which lie within the circles, we are left with $9\sqrt{2} - 3\sqrt{2} - \sqrt{2} = 5\sqrt{2}$. The correct answer is E.

This can also be proved by taking an arbitrary point on each of the circles, finding the distance between them and then minimising this, but that is far more complicated than needed.

Question 4

We can solve the equations to find x by substituting for y.

The second equation gives $y = a - x$, so the first equation becomes

$$3x^2 + 2x(a - x) = 4$$

This rearranges to

$$x^2 + 2ax - 4 = 0$$

and the discriminant of this equation is

$$(2a)^2 - 4 \times 1 \times (-4) = 4a^2 + 16$$

This is always positive, as $4a^2 \geqslant 0$ for all real values of a, so there are always two distinct real roots, and the correct answer is G.

Question 5

For this question, we make use of the remainder theorem: the remainder when $\mathrm{f}(x)$ is divided by $(x-a)$ is $\mathrm{f}(a)$.

In this case, we take $a=-2$ and $a=-3$, so

$$\mathrm{f}(-2) = R$$
$$\mathrm{f}(-3) = S$$

Substituting these into the given formula for $\mathrm{f}(x)$ gives

$$\mathrm{f}(-2) = -8 \quad + 4a - 2b + c = R$$
$$\mathrm{f}(-3) = -27 + 9a - 3b + c = S$$

We want the largest possible value of $R-S$, so we subtract these two equations to get

$$19 - 5a + b = R - S$$

To get the largest possible value, we want a to be as small as possible and b to be as large as possible, so we take $a=1$, $b=3$ (and then $c=2$), giving $R-S = 19-5+3 = 17$

Thus the correct answer is D.

Question 6

This equation has trigonometric functions and an x on its own. Our best bet is to first rearrange it to bring all of the trigonometric functions together. We have a $\sin 2x$ on one side and a $\cos 2x$ on the other, which suggests that we should divide by $\cos 2x$ to obtain $\tan 2x$. This gives

$$x \tan 2x = 1$$

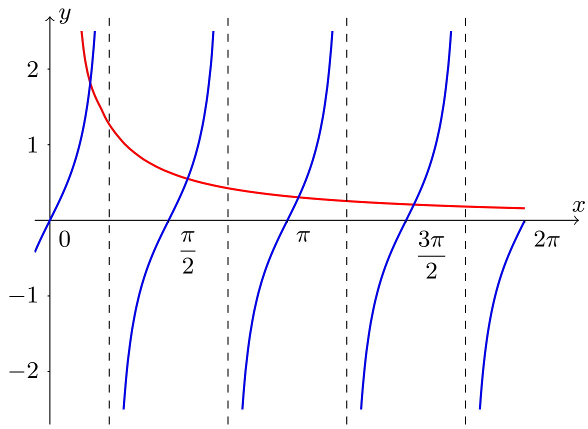
We should check, though, that we cannot have $\cos 2x = 0$, so that we are allowed to divide by it. If $\cos 2x = 0$, this would require $x \sin 2x = 0$, so either $\sin 2x = 0$, which is impossible if $\cos 2x = 0$, or $x = 0$. But if $x = 0$, then $\cos 2x = 1$, so this is also impossible. So we cannot have $\cos 2x = 0$.

Returning to $x \tan 2x = 1$, we can divide by x to get

$$\tan 2x = \frac{1}{x}$$

This equation is not possible to solve exactly. But the question (thankfully) only asks us how many solutions there are in the range $0 \leqslant x \leqslant 2\pi$

So we can sketch the graphs of $y = \tan 2x$ and $y = \frac{1}{x}$ and see how many times they intersect.



We see that the graph of $y = \frac{1}{x}$ intersects every positive branch of $y = \tan 2x$ once. Since there are four branches of $y = \tan 2x$ in the range $0 \leqslant x \leqslant 2\pi$, there are four solutions, so the correct answer is E.

Question 7

We can expand these two expressions using the binomial theorem. We look just at the term involving x^6:

$$\begin{aligned}(1+kx^2)^7 &= \cdots + \binom{7}{3}1^4(kx^2)^3 + \cdots \\ &= \cdots + \frac{7\times 6\times 5}{3\times 2\times 1}k^3x^6 + \cdots \\ &= \cdots + 35k^3x^6 + \cdots\end{aligned}$$

and

$$\begin{aligned}(k+x)^{10} &= \cdots + \binom{10}{6}k^4x^6 + \cdots \\ &= \cdots + \binom{10}{4}k^4x^6 + \cdots \\ &= \cdots + \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}k^4x^6 + \cdots \\ &= \cdots + 210k^4x^6 + \cdots\end{aligned}$$

We therefore need

$$35k^3 = 210k^4$$

so $k = \dfrac{35}{210} = \dfrac{5}{30} = \dfrac{1}{6}$.

Thus the correct answer is A.

Question 8

Let us write the terms of the GP (geometric progression) as a, ar, ar^2 and so on. Then we are told that

$$S_\infty = a + ar + ar^2 + \cdots = \frac{a}{1-r} = 6 \tag{1}$$

We can rearrange this final equation to $a = 6 - 6r$ if we wish to.

The squares of each term are a^2, a^2r^2, a^2r^4, a^2r^6 and so on. These themselves form a GP with first term a^2 and common ratio r^2. Therefore, given that their sum is 12, we have for this GP

$$S_\infty = a^2 + a^2r^2 + a^2r^4 + \cdots = \frac{a^2}{1-r^2} = 12$$

We can write this last equation as $\dfrac{a^2}{(1+r)(1-r)} = 12$

But we can now substitute in equation (1) to give

$$6\left(\frac{a}{1+r}\right) = 12$$

so that $\dfrac{a}{1+r} = 2$, or $a = 2 + 2r$

Thus we have simultaneous equations for a and r:

$$a = 6 - 6r$$
$$a = 2 + 2r$$

Subtracting these gives $4 - 8r = 0$, so $r = \frac{1}{2}$ and $a = 3$

We can now work out the sum to infinity of the cubes: the sequence of terms becomes a^3, a^3r^3, a^3r^6 and so on, which as before is a GP; this time the first term is a^3 and the common ratio is r^3. Hence the sum to infinity is

$$S_\infty = \frac{a^3}{1-r^3} = \frac{3^3}{1-(\frac{1}{2})^3} = \frac{27}{(\frac{7}{8})} = \frac{216}{7}$$

and the correct answer is D.

Question 9

> ***Commentary:*** *We want the cubic equation* $2x^3 - 3x^2 - 12x + c = 0$ *to have three distinct real solutions. We don't know any techniques for finding the roots of a cubic equation, so we will have to be creative. (Actually, there is a formula, often known as Cardano's formula, for solving a cubic equation, but you are not expected to know this.)*
>
> *What we* do *know how to do is to sketch a cubic graph by finding its turning points and* y*-axis intercept, even if we don't know the* x*-axis intercepts, so we will do this.*

We can find the stationary points on the graph of $y = 2x^3 - 3x^2 - 12x + c$ by differentiating. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

so there are stationary points at $x = 2$ and $x = -1$ (where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$)

These have coordinates

$$(-1, 7 + c) \quad \text{and} \quad (2, -20 + c)$$

For the equation $2x^3 - 3x^2 - 12x + c = 0$ to have three distinct real solutions, we need the two turning points of the cubic to be on opposite sides of the x-axis, one above it and one below it. It is clear that $-20 + c < 7 + c$, so we need $-20 + c < 0$ and $7 + c > 0$. The first inequality becomes $c < 20$ and the second $c > -7$.

Combining these, there are three distinct real solutions if and only if $-7 < c < 20$, which is option B.

Question 10

We notice that the first inequality involves x only and the second involves y only, so we can solve each inequality separately.

For the first one, $|2 - x| \leqslant 6$, this means that x is at distance at most 6 from 2, so $-4 \leqslant x \leqslant 8$

Likewise, the second inequality, $|y + 2| \leqslant 4$ means that y is at most 4 from -2 (as $|y + 2| = |y - (-2)|$, so it measures the distance of y from -2). So $-6 \leqslant y \leqslant 2$

We can now obtain the greatest possible value of $|xy| = |x|.|y|$ by taking the greatest possible value of $|x|$, which is 8, and the greatest possible value of $|y|$, which is 6, so the greatest possible value of $|xy|$ is $8 \times 6 = 48$, which is option E.

Question 11

We can find the equation of the normal to the curve $y = 10 - x^2$ at the point (p, q) as usual.

The derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$ so at $x = p$ the tangent has gradient $-2p$. Therefore the normal has gradient $\frac{1}{2p}$.

The tangent and normal both pass through the point $(p, 10 - p^2)$, which shows that $q = 10 - p^2$. The normal thus has equation

$$y - (10 - p^2) = \frac{1}{2p}(x - p)$$

so rearranging into the form $y = mx + c$ gives

$$y = \frac{1}{2p}x - \frac{1}{2} + 10 - p^2$$

We want this to have the form $y = mx + 5$ where $m > 0$, so we have

$$-\frac{1}{2} + 10 - p^2 = 5.$$

Rearranging thus gives

$$p^2 = \frac{9}{2}$$

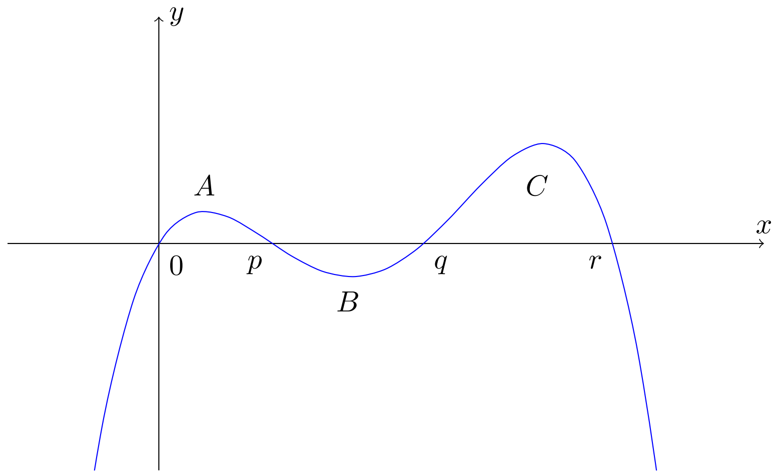
and hence $p = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

Since $m = \frac{1}{2p} > 0$ we must have $p > 0$ so $p = \frac{3\sqrt{2}}{2}$

The correct answer is C.

Question 12

This curve intercepts the x-axis at 0, p, q and r. As the leading power of x is $-x^4$, the curve tends to $-\infty$ as x gets very large (both positive and negative), and so the curve looks something like this:



We have labelled the three areas as A, B and C, and we want to know the value of $A+B+C$.

We can then write the three integrals we have been given in terms of these, referring to the sketch:

$$\int_0^r \mathrm{f}(x)\,\mathrm{d}x = A - B + C = 0 \qquad (1)$$

$$\int_0^q \mathrm{f}(x)\,\mathrm{d}x = A - B = -2 \qquad (2)$$

$$\int_p^r \mathrm{f}(x)\,\mathrm{d}x = -B + C = -3 \qquad (3)$$

Subtracting (3) from (1) gives $A = 3$, from which (2) gives $B = 5$ and so $C = 2$. Therefore $A + B + C = 10$, hence the correct option is F.

> ***Commentary:*** *Although our sketch turns out to be not very accurate, it was sufficient to help us understand what was going on and to solve the problem. This is what we seek from sketches: they should be good enough to give us insight, but they do not need to be perfect.*
>
> *Incidentally, there do exist values of p, q and r which make the statements of this question true, namely $p = 1.25$, $q = 3.16$ and $r = 4.25$, correct to two decimal places.*

Question 13

A local minimum of $f(x)$ occurs where the gradient is 0 at the point, is negative immediately to the left of the point and is positive immediately to its right.

So on the graph of $f'(x)$, this will be at a point where $f'(x) = 0$, and $f'(x)$ is increasing from negative to positive at that point. The only such point on the graph given is C.

The behaviour of $f(x)$ at all of the labelled points is as follows:

A Changing from positive to negative gradient, so a local maximum.

B A local minimum of the gradient, so a non-stationary point of inflection. If walking along the curve, one would stop turning to the right and start turning to the left.

C Changing from negative to positive gradient, so a local minimum.

D This point, where $x = 0$, has no particular relevance to the shape of $f(x)$.

E A local maximum of the gradient, so a non-stationary point of inflection. If walking along the curve, one would stop turning to the left and start turning to the right.

F A local minimum of the gradient at a point of zero gradient, so a stationary point of inflection. If walking along the curve, one would stop turning to the right and start turning to the left.

Question 14

There are a variety of different ways to approach this question. The following is probably the simplest.

We can substitute the coordinates of the points into the equation of the line to obtain

$$\begin{aligned} \log_2 p &= 3m + 4 \\ 4 &= m \log_2 p + 4 \end{aligned}$$

The second equation gives $m \log_2 p = 0$, so either $m = 0$ or $\log_2 p = 0$, so either $m = 0$ or $p = 1$.

If $m = 0$, then the first equation gives $\log_2 p = 4$, so $p = 16$

If $\log_2 p = 0$, so $p = 1$, then the first equation gives $3m + 4 = 0$, which has a solution.

So there are two possible values of p, namely 1 and 16. Hence the answer is B.

> ***Commentary:*** *You might find it helpful to write q (for example) in place of $\log_2 p$, so that the equations become*
>
> $$\begin{aligned} q &= 3m + 4 \\ 4 &= mq + 4 \end{aligned}$$
>
> *This makes their structure much more obvious.*

Question 15

We note that $3 = \left(\sqrt{3}\right)^2$.

So if we write $y = \left(\sqrt{3}\right)^x$, the equation becomes

$$y^2 - \left(\sqrt{3}\right)^4 y + 20 = 0$$

or

$$y^2 - 9y + 20 = 0$$

This factorises to give $(y-4)(y-5) = 0$ so either $y = 4$ or $y = 5$

We can write $y = \left(3^{\frac{1}{2}}\right)^x = 3^{\frac{1}{2}x}$ so either $3^{\frac{1}{2}x} = 4$ or $3^{\frac{1}{2}x} = 5$

We can now take logs to base 3 to obtain

$$\tfrac{1}{2}x = \log_3 4 \quad \text{or} \quad \tfrac{1}{2}x = \log_3 5$$

so

$$x = 2\log_3 4 \quad \text{or} \quad x = 2\log_3 5$$

The question has asked for the sum of the real solutions of the equation. Both of these are real, so we can add them to get $2\log_3 4 + 2\log_3 5 = 2\log_3 20$ using the log rule $\log ab = \log a + \log b$

Hence the answer is E.

Question 16

We first need to find the stationary point of the curve C. We can do this either by differentiating or, as the equation is a quadratic, by completing the square.

By differentiating, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + b$$

so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ if and only if $x = -\frac{1}{2}b$, and so

$$y = (-\tfrac{1}{2}b)^2 + b(-\tfrac{1}{2}b) + 2 = 2 - \tfrac{1}{4}b^2$$

Alternatively, we can complete the square to give $y = (x + \frac{1}{2}b)^2 - \frac{1}{4}b^2 + 2$, and so the stationary point is at $(-\frac{1}{2}b, 2 - \frac{1}{4}b^2)$.

The distance of this point from the origin is then given by

$$\sqrt{(-\tfrac{1}{2}b)^2 + (2 - \tfrac{1}{4}b^2)^2} = \sqrt{\tfrac{1}{4}b^2 + 4 - 4(\tfrac{1}{4}b^2) + \tfrac{1}{16}b^4} = \sqrt{\tfrac{1}{16}b^4 - \tfrac{3}{4}b^2 + 4}$$

To minimise this square root, we can minimise the expression inside the square root. This is a function of $B = b^2$, namely $\frac{1}{16}B^2 - \frac{3}{4}B + 4$, so we can minimise this.

> ***Commentary:*** *If a quartic is actually a quadratic in the variable squared, it is usually much easier to work with the quadratic than the quartic.*

We can minimise this by completing the square or differentiating, as before. We can start by taking out a factor of $\frac{1}{16}$ so we are trying to minimise $\frac{1}{16}(B^2 - 12B) + 4$. (There's no need to think about the 4 at the end: this additional term does not change the value of b or B at which the function is minimised.)

We therefore wish to minimise $B^2 - 12B = (B - 6)^2 - 36$ so $B = 6$ gives the minimum value, so $b^2 = 6$ giving $b = \sqrt{6}$ (as $b \geqslant 0$), hence the answer is F.

> ***Commentary:*** *We could have gone ahead and differentiated the whole square root expression for distance, if we know the chain rule. But that's more effort than needed. Similarly, we could have differentiated the expression as a quartic, but again that is more complex than required. It is almost always worth spending a moment thinking how to make the problem a little easier before diving into algebra and calculations.*

Question 17

It is not obvious how to begin this question – there are lots of wordy descriptions. We also don't know how many pieces of data there are in each of the two sets.

So let's try naming things.

Let the first set of data have n pieces of data, and the second set have m pieces of data.

Since the mean of the first set of data is 15, the sum of the data is $15n$.

Likewise, the sum of the second set of data is $20m$.

When we exchange one piece of data from the first set with one piece of data from the second set, the overall total stays the same, but the totals of the two sets change. We are told what happens to the means, so the total of the first set is now $16n$ and of the second set is $17m$.

Since the overall total is the same, we have

$$15n + 20m = 16n + 17m$$

We can simplify this to give $n = 3m$

Therefore the overall total is $15n + 20m = 45m + 20m = 65m$ and there are $n + m = 4m$ pieces of data.

Thus the mean of the combined set of data is $\dfrac{65m}{4m} = \dfrac{65}{4} = 16\frac{1}{4}$ and the answer is A.

> ***Commentary:*** *At the end of this, we are still no wiser about how many pieces of data there were to begin with. All we know is that the first set contains three times as many as the second set.*

Question 18

The lines of symmetry of a sine or cosine graph (that is, something of the form $\sin(kx + \theta) + c$ or $\cos(kx + \theta) + c$) are at the x-values where the sine or cosine equals ± 1. So in this case, if we solve $\sin(2x - \frac{4\pi}{3}) = \pm 1$ we will find the lines of symmetry.

We solve each of them separately.

We have $\sin\theta = 1$ when $\theta = \frac{\pi}{2} + 2n\pi$ (for any integer n), so in this case, this gives

$$\begin{aligned} & 2x - \tfrac{4\pi}{3} = \tfrac{\pi}{2} + 2n\pi \\ \iff \quad & 2x = \tfrac{11\pi}{6} + 2n\pi \\ \iff \quad & x = \tfrac{11\pi}{12} + n\pi \end{aligned}$$

So the smallest positive value of x with $\sin(2x - \frac{4\pi}{3}) = 1$ is $x = \frac{11\pi}{12}$

In the second case, we have $\sin\theta = -1$ when $\theta = -\frac{\pi}{2} + 2n\pi$ (for any integer n), so in this case, this gives

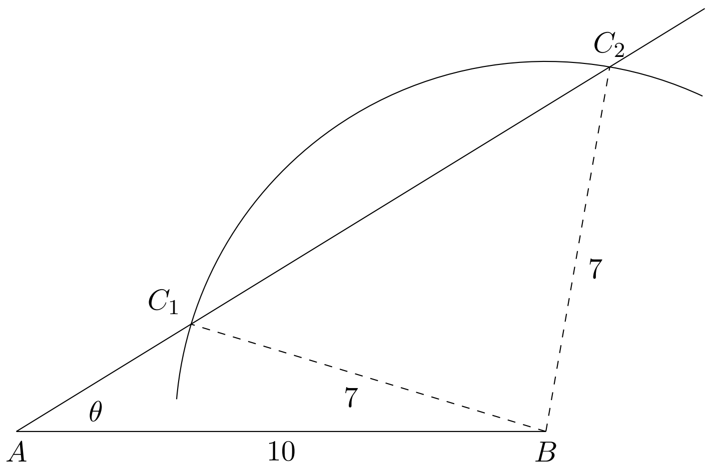
$$\begin{aligned} & 2x - \tfrac{4\pi}{3} = -\tfrac{\pi}{2} + 2n\pi \\ \iff \quad & 2x = \tfrac{5\pi}{6} + 2n\pi \\ \iff \quad & x = \tfrac{5\pi}{12} + n\pi \end{aligned}$$

So the smallest positive value of x with $\sin(2x - \frac{4\pi}{3}) = -1$ is $x = \frac{5\pi}{12}$

This is smaller than $\frac{11\pi}{12}$ so the answer is $\frac{5\pi}{12}$, which is option B.

Question 19

This questions calls for a sketch. We draw $AB = 10$, a line from A at an angle θ to AB (though we do not know what θ is), and a circular arc centred on B with radius 7.



The sketch shows the two triangles which could be formed subject to these conditions, with the third vertex being at C_1 or C_2.

One approach, hinted at by the instruction to find $\cos\theta$, is to use the cosine rule.

In the triangle ABC_1, we have

$$7^2 = (AC_1)^2 + 10^2 - 2(AC_1)10\cos\theta \tag{1}$$

while in the triangle ABC_2, we have

$$7^2 = (AC_2)^2 + 10^2 - 2(AC_2)10\cos\theta \tag{2}$$

Unfortunately, we do not know either AC_1 or AC_2. We do, however, know that the larger triangle has three times the area of the smaller one. Taking AC_1 or AC_2 as the base of the triangle, the perpendicular height is from B to the line AC_2, and is the same for both triangles. So $AC_2 = 3(AC_1)$

We can now substitute this into equation (2) to get

$$7^2 = 9(AC_1)^2 + 10^2 - 6(AC_1)10\cos\theta$$

Subtracting equation (1) then gives

$$0 = 8(AC_1)^2 - 40(AC_1)\cos\theta$$

so dividing by $8AC_1$, we have $AC_1 = 5\cos\theta$.

We can now substitute this back into equation (1) to get

$$7^2 = (5\cos\theta)^2 + 10^2 - 20(5\cos\theta)\cos\theta$$

so

$$49 = 25\cos^2\theta + 100 - 100\cos^2\theta$$

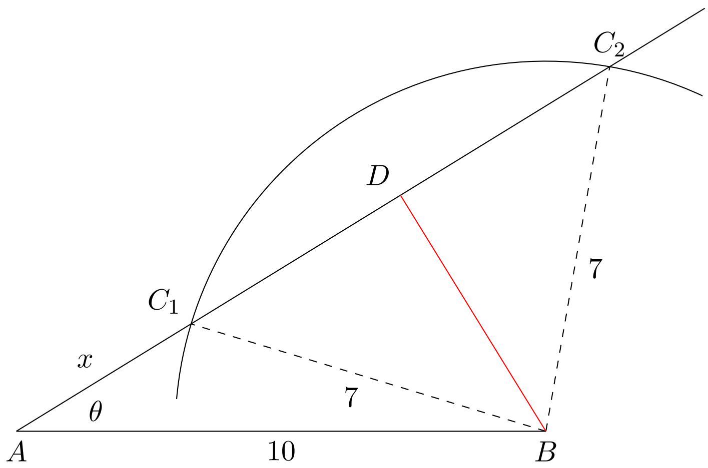
which simplifies to

$$75\cos^2\theta = 51$$

so $\cos^2\theta = \frac{51}{75} = \frac{17}{25}$ giving $\cos\theta = \frac{\sqrt{17}}{5}$ (as $\cos\theta > 0$)

There is an alternative approach which does not use the cosine rule.

If we drop a perpendicular from B to the line AC_2, and let $x = AC_1$, our diagram becomes:



Now D is midway between C_1 and C_2 by symmetry (or by using the "RHS" congruency rule on the triangles BDC_1 and BDC_2), so $C_1D = DC_2$.

Also, the area of the triangle ABC_1 is $\frac{1}{2}(AC_1)(BD)$ and the area of the triangle ABC_2 is $\frac{1}{2}(AC_2)(BD)$

Since the larger triangle has three times the area of the smaller one, $AC_2 = 3(AC_1)$ (as before). So $AC_2 = 3x$, hence $C_1C_2 = 2x$, and $C_1D = DC_2 = x$

We can now apply Pythagoras to BDC_1 and BDA. We obtain, respectively,

$$x^2 + (BD)^2 = 7^2$$
$$(2x)^2 + (BD)^2 = 10^2$$

Subtracting gives $3x^2 = 51$, so $x^2 = 17$

Since $\cos\theta = \frac{AD}{AB} = \frac{2x}{10}$ we get $\cos\theta = \frac{\sqrt{17}}{5}$ as before.

This also easily shows that $BD = \sqrt{32} = 4\sqrt{2}$ and that the area of the two triangles ABC_1 and ABC_2 are $2\sqrt{34}$ and $6\sqrt{34}$.

Question 20

This reminds us of adding the terms of an arithmetic series. One way to do so is to write out the series twice, once forwards and once backwards, and then add corresponding pairs of terms together. If we do that here, we obtain

$$S = \sin^2 0^\circ + \sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 88^\circ + \sin^2 89^\circ + \sin^2 90^\circ$$
$$S = \sin^2 90^\circ + \sin^2 89^\circ + \sin^2 88^\circ + \cdots + \sin^2 2^\circ + \sin^2 1^\circ + \sin^2 0^\circ$$

so that

$$2S = (\sin^2 0^\circ + \sin^2 90^\circ) + (\sin^2 1^\circ + \sin^2 89^\circ) + \cdots + (\sin^2 90^\circ + \sin^2 0^\circ)$$

Now we know that $\sin^2 x + \cos^2 x = 1$ for all x, and since $\sin(90^\circ - x) = \cos x$ (by, for example, thinking about a right-angled triangle or by looking at the graphs of the functions), we can rewrite this sum as

$$\begin{aligned} 2S &= (\sin^2 0^\circ + \cos^2 0^\circ) + (\sin^2 1^\circ + \cos^2 1^\circ) + \cdots + (\sin^2 90^\circ + \cos^2 90^\circ) \\ &= 1 + 1 + \cdots + 1 \\ &= 91 \end{aligned}$$

(since there are 91 terms), therefore $S = 45.5$, which is option E.

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